Uniwersytet Wrocławski

"Scalar field dynamics in black hole backgrounds"

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The aim of the project

What will happen to a black hole when it is only slightly perturbed by external field?

Bell is a good analogy. A perturbed black hole (by external field) will go through the following stages:

1. transient
2. quasi-normal mode ringdown
3. exponential or power law tail

The aim is to calculate these quasi-normal modes.

Presentation plan

1. Brief introduction to General Relativity.

2. Klein-Gordon equation in Schwarzschild metric.

3. Calculating black hole quasinormal modes by using Pöschl-Teller potential **(approximation)**.

4. Calculating black hole quasinormal modes by using confluent Heunn equation **(exact numerical solution)**

Brief introduction to General Relativity

Minkowski spacetime (flat spacetime):

$$\mathbf{x} = [x^0, x^1, x^2, x^3] \equiv x^\mu \mathbf{e}_\mu$$
$$\mathbf{y} = [y^0, y^1, y^2, y^3] \equiv y^\nu \mathbf{e}_\nu$$

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{\mu,\nu=0}^{3} \eta_{\mu\nu} x^\mu y^\nu$$

Distance beetween $(\mathbf{x} + d\mathbf{x})^\mu$ and $\mathbf{x}^\mu$ (metric):

$$d\mathbf{s}^2 = \eta_{\mu\nu} d\mathbf{x}^\mu d\mathbf{x}^\nu = -(cdt)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2$$

Curved spacetime – Einstein field equation:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu}$$

"Spacetime tells matter how to move; matter tells spacetime how to curve." ~ J. A. Wheeler

$R_{\mu\nu}$ - Ricci tensor (contracted Riemann curvature tensor)

$g_{\mu\nu}$ - Metric tensor

$T_{\mu\nu}$ - gravitational energy–momentum tensor, a source of the gravitational field (gravitating matter)

Vacuum field equations:

$$T_{\mu\nu} = 0 \Rightarrow R_{\mu\nu} = 0$$

Examples of solutions:
-Minkowski spacetime (flat spacetime)
-Kerr Solution
-Schwarzschild solution

Schwarzschild solution in spherical coordinates: $\left(x^0, x^1, x^2, x^3\right) = (t, r, \theta, \varphi)$

For a gravitational field outside a spherical mass, on the assumption that the electric charge of the mass and angular momentum of the mass are zero

Metric (*G* = *c* = *1*):

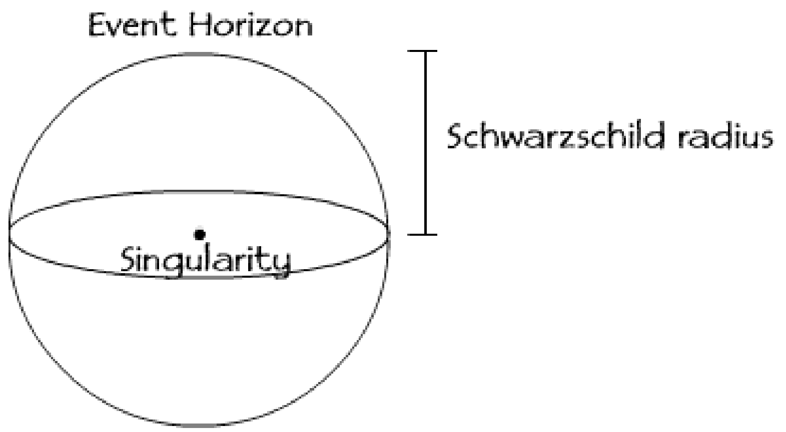
$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2$$

$$f(r) = 1 - \frac{2M}{r}$$

Shwarzschild radius:

$$r_s = 2M$$

Black holes:



Klein-Gordon equation in Schwarzschild metric

- scattering of a scalar field on a Schwarzschild background

Klein-Gordon equation in flat spacetime:

$$\left(\partial^2 + m^2\right)\psi = 0$$

Massless Klein-Gordon equation in curved spacetime:

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(g^{\mu\nu}\sqrt{-g}\partial_\nu\psi\right) = 0$$

We can obtain a wave equation not only for a scalar field, but also for electromagnetic and gravitational fields, just by adding a small perturbation to a fixed background spacetime metric (linearising GR):

$$g_{\mu\nu} = g^0_{\mu\nu} + \delta g_{\mu\nu}$$

$$\delta R_{\mu\nu} = 0$$

Massless Klein-Gordon equation in Schwarzschild spacetime (curved of course):

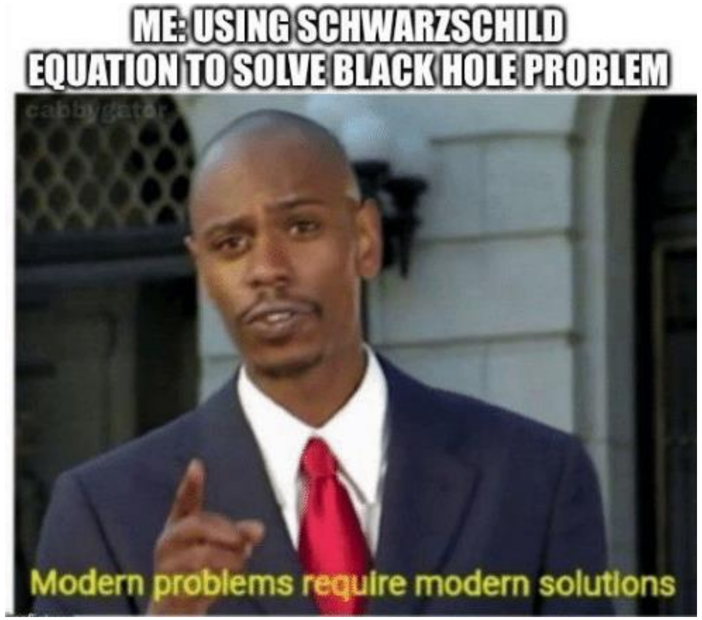
$$-f^{-1}\partial_t^2\psi + \tfrac{1}{r^2}\left[\partial_r\left(fr^2\partial_r\psi\right)\right] + \frac{1}{r^2\sin\theta}\left[\partial_\theta\left(\sin\theta\partial_\theta\psi\right)\right] + \frac{1}{r^2\sin^2\theta}\partial_\phi^2\psi = 0$$

The ansatz

$$\psi(t,r,\theta,\phi) = \sum_{\ell,m} e^{-i\omega t} Y_{\ell m}(\theta,\phi)\frac{\varphi(r)}{r}$$

which gives the following radial equation for $\varphi(r)$:

$$\omega^2\varphi + \frac{f}{r}\left[\partial_r\left(fr^2\partial_r\left(\frac{\varphi}{r}\right)\right)\right] - f\frac{\ell(\ell+1)}{r^2}\varphi = 0$$



To get rid off first order derivative, let's introduce *tortoise coordinate:*

$$\frac{dr_*}{dr} = \frac{1}{f}$$

$$r_* = r + 2M\ln(r - 2M)$$

$r \to \infty$ then $r_* \to \infty$

$r \to 2M$ then $r_* \to -\infty$

So we obtain radial stationary Regge-Wheeler equation in *tortoise coordinates:*

$$\left(\partial^2_{r*} - V(r) + \omega^2\right)\varphi = 0$$

$$V(r) = \left(1 - \frac{2M}{r}\right)\left[\frac{\ell(\ell+1)}{r^2} + \frac{2M}{r^3}\right]$$

This 'potential' can be generalised for other fields (not only scalar):

$$V_\ell(r) = \left(1 - \frac{2M}{r}\right)\left[\frac{\ell(\ell+1)}{r^2} + \frac{2M\,(1-s^2)}{r^3}\right]$$

Where:

$s = 0$ for scalar field

$s = 1$ for electromagnetic field

$s = 2$ for axial gravitational field

Quasi-normal modes

Once we have the wave equation, it needs to be solved with the appropriate boundary conditions. In order to produce the characteristic, free oscillations of the black hole spacetime, the wave solution must be:

-purely ingoing at the horizon

$$r \to 2M \Longleftrightarrow r_* \to -\infty$$

$$\psi \propto e^{-i\omega(t+r_*)}$$

$$e^{-i\omega t} = e^{-i(\omega_R + i\omega_I)t} = e^{\omega_I t} \cos(\omega_R t + \phi)$$

-purely outgoing at the infinity

$$r \to \infty \Longleftrightarrow r_* \to \infty$$

$$\psi \propto e^{-i\omega(t-r_*)}$$

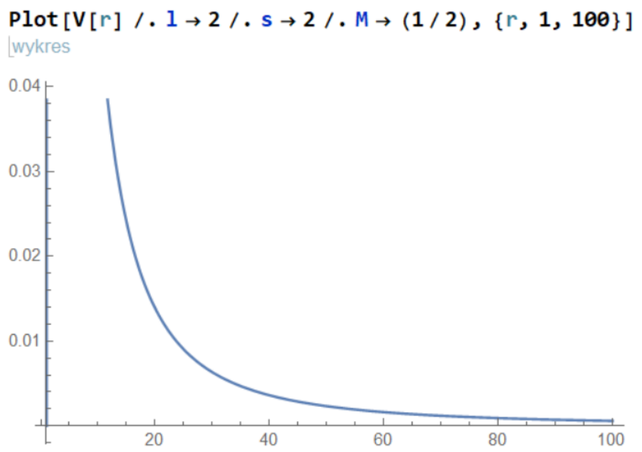
$\omega_I < 0$ exponential damping (stable)

$\omega_I > 0$ exponential growth (unstable)

Pöschl–Teller potential (approximation)

The Pöschl–Teller potential looks simillar to the potential from our radial equation:

$$V_{PT} = \frac{V_0}{\cosh^2 \alpha \left(r_* - r_*^0\right)} \qquad V(r) = \left(1 - \frac{2M}{r}\right)\left[\frac{\ell(\ell+1)}{r^2} + \frac{2M}{r^3}\right]$$



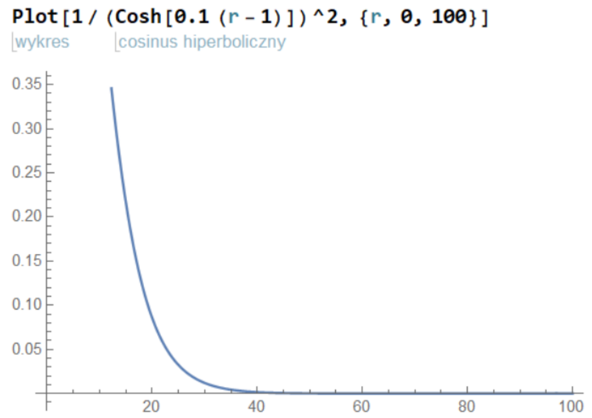
r_*^0 - is the point at which the potential attains a maximum

V_0 - is a value of the potential at that point

$$\alpha^2 \equiv -\left(2V_0\right)^{-1} d^2V/dr_*^2 \left(r_*^0\right)$$

Indeed, we can fit Pöschl–Teller potential and we start with equation:

$$\frac{\partial^2 \varphi}{\partial r_*^2} + \left(\omega^2 - \frac{V_0}{\cosh^2 \alpha \left(r_* - \overline{r}_*\right)}\right)\varphi = 0$$



We change variables:

$$\xi = \left[1 + e^{-2\alpha(r_* - \overline{r}_*)}\right]^{-1}$$

$$r \to \infty \Leftrightarrow r_* \to \infty \Leftrightarrow \xi \to 1$$

$$r \to 2M \Leftrightarrow r_* \to -\infty \Leftrightarrow \xi \to 0$$

$$\varphi = (\xi(1-\xi))^{-i\omega/(2\alpha)}$$

And we and up with hypergeometric equation:

$$\xi(1-\xi)\partial_\xi^2 y + [c - (a+b+1)\xi]\partial_\xi y - aby = 0$$

$$a = \left[\alpha + \sqrt{\alpha^2 - 4V_0} - 2i\omega\right]/(2\alpha)$$

$$b = \left[\alpha - \sqrt{\alpha^2 - 4V_0} - 2i\omega\right]/(2\alpha)$$

$$c = 1 - i\omega/\alpha$$

Whose solution around the horizon is just:

$$\varphi = A\xi^{i\omega/(2\alpha)}(1-\xi)^{-i\omega/(2\alpha)}F(a-c+1, b-c+1, 2-c, \xi)$$
$$+ B(\xi(1-\xi))^{-i\omega/(2\alpha)}F(a, b, c, \xi)$$

From boundary conditions, at the horizon we have: $A = 0$

To get a solution at the infinity, we need to know hypergeometric function when $\xi \to 1$

$$F(a,b,c,z) = (1-z)^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}F(c-a, c-b, c-a-b+1, 1-z)$$
$$+ \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}F(a, b, -c+a+b+1, 1-z)$$

From which, due to boundary conditions, we get:

$$1/\Gamma(a) = 0$$

$$1/\Gamma(b) = 0$$

These implies:

$$\omega = \pm\sqrt{V_0 - \alpha^2/4} - i\alpha(2n+1)/2\,, \quad n = 0,\, 1,\, 2,\, ...$$

Quasinormal modes of a black hole

Confluent Heunn equation (exact numerical solution)

We can do a substitution in Regge-Wheeler equation:

$$\left(\partial_{r*}^2 - V(r) + \omega^2\right)\varphi = 0 \Longleftrightarrow \partial_t^2\Psi + \left(-\partial_{r*}^2\Psi + V(r)\right) = 0$$

$$\Psi(r,t) = \psi(r)e^{i\omega t}$$

Consider the ansatz:

$$2M = 1$$

$$\varphi(r) = r^{s+1}(r-1)^{i\omega}e^{i\omega r}H(r)$$

From this, we obtain confluent Heunn equation:

$$\frac{\mathrm{d}^2\mathrm{H}}{\mathrm{d}r^2} + \left(\frac{\gamma}{r} + \frac{\delta}{r-1} + \nu\right)\frac{\mathrm{dH}}{\mathrm{d}r} + \frac{\alpha\nu r - \sigma}{r(r-1)}\mathrm{H} = 0$$

$$\gamma = 1+2s, \qquad \delta = 1+2i\omega, \qquad \nu = 2i\omega, \qquad \sigma = -(s-\ell)(s+1+\ell), \qquad \alpha = s+1$$

Local solutions for confluent Heun equation

1. Local solution around the horizon (r = 1):

$$\Psi_{\pm}^{(1)}(t,r) = e^{i\omega t}\psi_{\pm}^{(1)}(r)$$

$$\psi_{\pm}^{(1)}(r) = r^{s+1}e^{i\omega r \pm i\omega\log(1-r)}\mathrm{HeunC}(-2i\omega, \pm 2i\omega, 2s, -2\omega^2, 2\omega^2+s^2-\ell(\ell+1), 1-r)$$

2. Asymptotic solution around the spatial infinity:

$$\Psi_{\pm}^{(\infty)}(t,r) = e^{i\omega t}\psi_{\pm}^{(\infty)}(r)$$

$$\psi_{\pm}^{(\infty)}(r) \sim e^{\mp i\omega r - i\omega\log r}\sum_{\nu\geq 0}\frac{a_{\nu}^{\pm}}{r^{\nu}}, \qquad a_0^{\pm} = 1$$

3. Different local solutions are related as follow:

$$(1\leftarrow) = \Gamma_{\infty+}^{1+}(\rightarrow\infty) + \cancel{\Gamma_{\infty-}^{1+}(\leftarrow\infty)}$$ then $$\Gamma_{\infty-}^{1+}(\omega, s, l) = 0 \Rightarrow \omega = \omega(n, s, l), n = 0, 1, 2\ldots$$

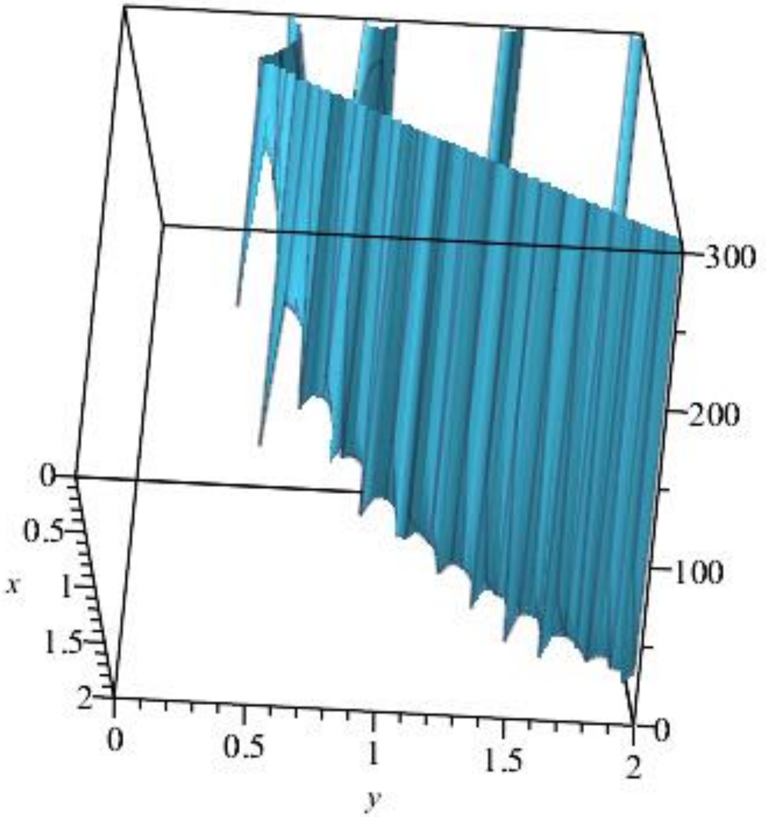
It happens if and only if:

$$\lim_{|r|\to+\infty}\left|r^{s+1}\,\mathrm{HeunC}\left(-2i\omega, 2i\omega, 2s, -2\omega^2, 2\omega^2+s^2-l(l+1), 1-r_{\infty}e^{i\left(\frac{\pi}{2}+\arg(\omega)\right)}\right)\right| = 0$$

Numerical solution

$$\left|(r_\infty)^{s+1}\,\mathrm{HeunC}\left(-2i\omega, 2i\omega, 2s, -2\omega^2, 2\omega^2+s^2-l(l+1), 1-r_\infty e^{i\left(\frac{\pi}{2}+\arg(\omega)\right)}\right)\right| = 0$$

Let $\quad l=2, s=2, r_\infty=20$



$n=0$	0.747343368 + i 0.177924631
$n=1$	0.693421994 + i 0.547829750
$n=2$	0.602106909 + i 0.956553966
$n=3$	0.503009924 + i 1.410296405
$n=4$	0.415029159 + i 1.893689781

Conclusions

1. We calculated quasi-normal modes of a Schwarzschild Black Hole peturbated by external fields. We have obtained approximated result and exact numerical result.

2. It is possible to do these calculations for diffrent metrics, for example Kerr metric (rotating black hole).

3. There is a need for new mathematical methods in order to find fully analitical solutions (for instance tools of 2d CFT).

Bibliography

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Thank you for your attention